Classical Tunneling from the Lorentz-Dirac Equation

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Abstract

The classical equation of motion of a charged point particle, including its radiation reaction, implies tunneling. For nonrelativistic electrons and a square barrier, the solution is elementary and explicit. We show the persistance of the solution for smoother potentials. For a large range of initial velocities, initial conditions may leave a (discrete) ambiguity on the resulting motion.

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In classical relativistic electrodynamics the motion of an electrically charged point particle (without further structure; we will call it an 'electron' for brevity.) is governed by the Lorentz-Dirac (LD) equation [1]. For general reference, see [2], and especially [3]. It includes the effect of the back-reaction of the (retarded) field generated by the electron during its past on its own motion. The radiation reaction is taken into account through a renormalisation procedure in which the bare mass of the electron and the electromagnetic self-energy combine to the physical inertial mass. We will show that this classical equation exhibits tunneling.

There exists a variety of methods [1,3–8] to derive this equation, the most comprehensive ones being based on energy and momentum conservation. The LD equation emerges as the unique result, if one assumes that, except its charge, the electron possesses no other attributes like dipole or higher multipole moments. It reads

$$\ddot{z}^\mu = \frac{e}{m} F^{\mu\nu} \dot{z}_\nu + \tau \left(\frac{\ddot{z}^2 \dot{z}^\mu}{c^2} + \dddot{z}^\mu \right) \tag{1}$$

where z^μ is the position of the electron (and the singularity of its electromagnetic field), dots denote derivatives with respect to proper time, and $\tau = \frac{2}{3} \frac{e^2}{4\pi\epsilon_0 mc^3} \simeq 0.62 \ 10^{-23}$ s is the 'pre-acceleration time'. The Lorentz force term $F^{\mu\nu}$ only includes the electromagnetic field generated by external sources. The last two terms in eq.(1) are due to the radiation reaction. The first is (minus) the derivative of the radiated four momentum, and the second, the Schott term, can be combined with the left hand side into the time rate of change of $p_\mu = m(\dot{z}_\mu - \tau \ddot{z}_\mu)$, the "bound momentum". That momentum can consistently be interpreted as the total momentum of the electron together with its bound field [8]. We will simply call it the momentum in the sequel. Asymptotically, when the acceleration ceases, it is equal to the usual momentum.

To set in perspective the remarkable fact that this *classical* equation allows the electron to tunnel through potential barriers that are narrow enough to be crossed in a proper time τ, we first discuss some general features of eq.(1). Although the Lorentz-Dirac equation describes the radiation reaction in a satisfactory way, and has been used in a variety of circumstances – the most extreme ones being the astrophysical applications [9] – it also exhibits some features that have raised eyebrows. For this third order equation, the initial value problem demands a specification of initial position, velocity *and acceleration.* The solution is then unique, but in general unphysical, describing a 'runaway', i.e. a motion with an exponentially increasing velocity, even in a force-free spatial region. The specification of initial acceleration has therefore traditionally been replaced by an *asymptotic* condition[1] stating that the (unobserved) runaways are rejected. From the mathematical point of view, the first question to be answered is then, whether a satisfactory solution still exists for reasonable initial data, and whether it is unique. This last point is "one of the most important unsolved problems of the theory" [10], and remained unresolved, even though some isolated cases have been found [11] of initial data (for position and velocity) that allow more than one solution. The LD-equation also gives rise to a new physical phenomenon: pre-acceleration. Indeed, the effect of the asymptotic boundary condition in the future makes

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[1]or, perhaps more physically, by formulating the LD equation as an integro-differential equation.

itself felt at early times, implying that an electron approaching a sharply delineated region in space where a force field is present, actually starts accelerating *before* it reaches the force field. This is sometimes viewed upon as an undesirable feature. A number of alternatives to the LD equation have been (re-)proposed over the years [12], but either run into difficulties (mainly with energy conservation), or necessitate additional structure. Although some of them may be valid as models for an electron that has an extended structure, we will not consider these alternatives further.

In view of the considerable attention that the LD equation, and its solutions, have generated, it is remarkable that the class of solutions that we are about to describe, with surprising physical implications, has gone unnoticed so far. They describe *tunneling*. Explicit examples can be found by considering the familiar setting from quantum mechanics: a one dimensional problem, where an electron impinges on a region with a rectangular potential energy barrier. For our purposes the nonrelativistic approximation (NRA) will be sufficient. Let us denote the electrostatic field as $F = -dV/dx$. Choosing units such that the electron mass, the velocity of light, and the characteristic time τ are equal to unity, the equation becomes, with $v = \dot{x}$,

$$\dot{p} = \dot{v} - \ddot{v} = F. \tag{2}$$

The electron only experiences a force when crossing the boundaries of the regions of constant potential. The solutions in the separate force-free regions, which are easy to write down explicitly, are connected using the following matching condition[2] on the momentum, or the acceleration, the position and the velocity being continuous:

$$\Delta p = -\Delta \dot{v} = -\Delta V / v . \tag{3}$$

The asymptotic condition is most easily implemented by solving the matching conditions backward in time, putting $\dot{v}_f = 0$ (a procedure that is also expedient when numerically integrating the equation). In the non-relativistic approximation eq.(2), this results in the following set of equations relating the initial and final velocities to the time T spent in the barrier region of width w and height V:

$$w = v_f T - \frac{V}{v_f}(e^{-T} - 1 + T) ,$$

$$v_i = v_f - \frac{V}{v_f} + \frac{V}{v_f - \frac{V}{v_f}(1 - e^{-T})} . \tag{4}$$

Although the analysis of these equations in general is not very difficult, it becomes particularly simple when the final electron energy is equal to half the barrier height, $V = v_f^2$. An explicit example is, for $V = 144, w = 3$:

$$\begin{aligned} x = \quad &-7(e^t - 1) + 16t & t < 0 , \\ &9(e^t - 1) & 0 < t < T , \\ &3 + 12(t - T) & T < t , \end{aligned} \tag{5}$$

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[2]This can be checked using the formal equation $\dot{p} = \Delta V \cdot \delta(x)$. The validity of the rectangular barrier idealization will be discussed shortly.

with $T = \log 4/3$. The matching condition eq.(3) implies jumps of 16 and -12 units in the acceleration at $t = 0$ and $t = T$. Tunneling occurs, the initial energy is equal to 128, a fraction 1/9 below the barrier height.

Whereas the details of the example above are of course special, the tunneling phenomenon is actually quite generic. A decisive parameter is the width of the potential. If the electron can cross the barrier within a time of order 1, i.e. the pre-acceleration time τ, tunneling occurs. To understand this it suffices to follow the bound momentum while the electron crosses the barrier. Its value is piecewise constant, and in the regions outside the barrier equal to its asymptotic value. Under the barrier itself the value is $v_f - V/v_f$, which may be in the same direction *or opposite* to the velocity. The example above is special, in that the intermediate momentum vanishes. Without external force, if the velocity is directed oppositely, it will quickly turn in a time of order τ in the same direction as the (conserved) momentum. If the width is small enough, the electron has in the meantime reached the opposite side of the barrier. The smaller the width, the wider the range of initial velocities for which the electron will tunnel through.

The explicit solution given above can be used to illustrate another remarkable property of the LD equation (together with the asymptotic condition), viz. the failure of initial data to determine the solution uniquely. If we take a barrier extending from $x = 3$ to some value $x < -9$, and specify $x = -9$ and zero velocity in the infinite past, the second line of eq.(5), extended to negative times, together with the last line, constitute a solution that is an alternative to the trivial one with constant x. This nonuniqueness is also mentioned in [11]. We do not regard this as a sufficient answer to the initial condition question of Rohrlich [10] cited above. It is analogous to the ambiguity present already in Newtonian mechanics, when specifying, in the infinite past, a zero velocity at the top of a mountain: it is an isolated special case, and an infinitesimally small initial velocity eliminates the stationary solution. Of considerably more interest is the fact that rectangular barrier crossing, whether by tunneling through or by passing over it, very often exhibits a much more generic non-uniqueness[3]. For values of the initial velocity sufficiently large to cross the barrier, there is in general more than one distinct solution, typically one where almost all energy is radiated away and up to five (see later) where a smaller radiation loss occurs. This shows that the non-uniqueness is a common feature of the LD equation: given the no-runaway condition, it is generically still insufficient to specify position and momentum at some initial time. This is true not only for asymptotic initial values, but also for initial values at finite times.

It is important to realize that the features discussed above do *not* depend on the non-relativistic approximation involved. There is a simple scaling property of the NRA that leaves this equation invariant, viz. rescaling all lengths, velocities and accelerations with a common factor, and the barrier height with its square. This implies that we can always rescale such that only small velocities are involved, and our discussion applies. The existence of the tunneling solutions is therefore beyond doubt.

What is less clear, is the possible role played by the sharpness of the potential step. Indeed, for the discontinuous potential step, it turns out that the problem as formulated

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[3]A similar phenomenon has been noticed also in [13]. However, that study has some problems, see below.
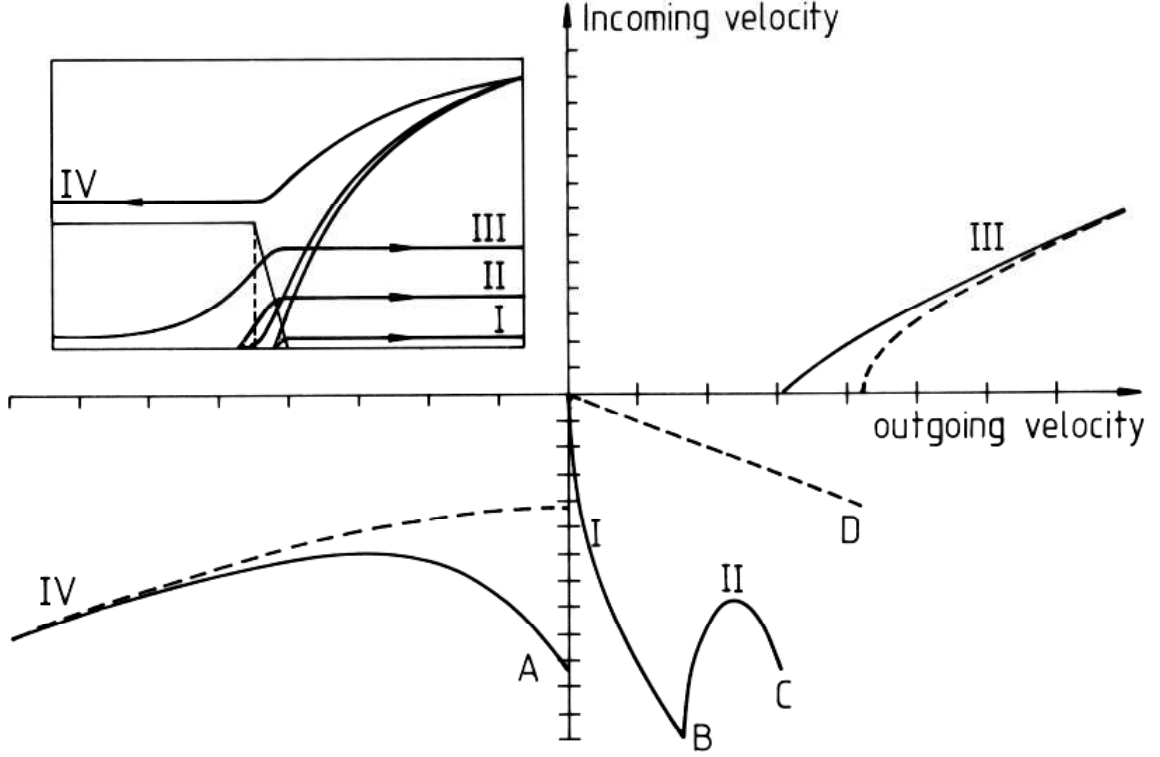
FIG. 1. Plot of the initial velocity vs. the final velocity for the solution of the Lorentz-Dirac equation in a linearly rising step potential (note the difference in scale). The dotted lines leave out the radiation reaction. The inset shows the potential and the kinetic energy. The four types of motion are discussed in the text. For the plots, a step height $V = 9$ was used, and a slope width $\epsilon = 0.5$.

above has *no* solution for small incident velocities. This was noticed, for a single potential step, in [13]. It was stated there that no mathematical difficulties are associated with the idealization of a sharp step. Physically, however, this absence of solutions for a given initial velocity range is probably unacceptable. To investigate this point, and to ascertain at the same time that the tunneling feature is not correlated to this unphysical behavior, we have repeated the analysis when the electron climbs a ramp with a finite slope. We take the force F to be a constant over a region of width ϵ, with[4] $V = \epsilon F$. At the moment we consider only a single ramp, we will come back to the tunneling situation later. Again, in each of the regions of constant force, the LD equation can be solved exactly (NRA) in terms of elementary functions, for instance in the sloping region $x = -F(e^t - 1 - t - t^2/2) + v_f t$. Then we again investigate and the matching conditions. We use the same backward-in-time method as before.

In figure 1 (see the inset), we have plotted the kinetic energy as a function of the position for typical cases. The four different possibilities that arise are listed in table I.

Figure 1 also shows a plot of the initial versus the final velocity. The following remarks

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[4]We will always consider a fixed height V when we consider the small ϵ limit.

TABLE I. Overview of the four different types of motion, described in the text.

case	v_i	Turning point	v_f
I	$-$	inside sloping region	$+$
II	$-$	under plateau	$+$
III	$+$		$+$
IV	$-$		$-$

can be made.

Case III: This motion agrees fully with intuition. For a gentle slope ($\epsilon \to \infty$) the radiation energy loss is negligible, so that the final velocity is always larger than $\sqrt{2V}$; for a steep slope ($\epsilon \to 0$) with the same height, and small initial velocity, half the energy is radiated away due to the larger acceleration, and the final velocity approaches $\sqrt{V}$.

Case IV: Here it is of course necessary that the electron has enough energy ($> V$) to overcome the barrier, and if the slope is gentle this is sufficient. For a steep slope the minimum initial energy is $2V$. Note however the surprising feature that the solution is not fixed by the initial velocity alone: for a range of initial velocities larger than the minimum required to overcome the barrier, there are actually two different solutions. This range becomes larger as the barrier gets steeper, as $6^{1/3}V^{2/3}\epsilon^{-1/3}$ (point A in the figure). So we see now that nonuniqueness of solutions indeed persists when the potential step is replaced with a slope, and in fact we have also checked it numerically for a completely smooth ramp (a hyperbolic tangent).

Cases I and II: Both represent reflections. The branch starting at the origin corresponds to case I, with $\epsilon \to 0$ limiting behavior $v_f \sim v_i^2\epsilon/6V$. The point B where case II takes over is located at $v_i \sim 3^{5/6}V^{2/3}\epsilon^{-1/3}$. This shows that the behavior for a steep ramp is quite subtle. If one investigates only the (formal) limiting equation without taking this into account, one is likely to miss branch I [5]although it is clearly a physically correct possibility, and in fact, for small velocities, this solution is unique. In the small ϵ limit, this branch tends to the vertical axis. The type II branch on the other hand has a smoother limit, and reduces to the straightforward solution for infinite slope, as obtained from the matching condition eq.(3). There is no type II solution beyond point C in the figure: electrons with a larger final velocity necessarily originate from the plateau, and are shown on branch III. For a very gentle slope, the whole compound curve I and II, will approach the line $v_i = -v_f$ representing no radiation loss, while both points B and C converge to D ($v_f = \sqrt{2V}$). For a very steep slope, both points B and C move towards infinite initial velocities as $\epsilon^{-1/3}$ (with a fixed ratio $\sqrt{3}/\sqrt[3]{2}$), and limiting final velocities equal to 0 and $\sqrt{V}$ respectively.

Thus, when a high velocity electron meets a very steep well, there is an amazing variety of different possible outcomes. It may lose some energy and travel on (curve IV, left branch), just barely make it up the hill (curve IV, right branch) having lost most of its energy in radiation, or be reflected with a choice of *three* different velocities!

It is clear that analogous results hold for a barrier with finite width, and that tunnel-

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[5]The numerical study in [13] exhibits this problem. We therefore reject the conclusion reached there, that for small initial velocities no solutions would exist at all.

ing solutions will persist for $\epsilon \neq 0$. Furthermore, we have checked numerically that the qualitative behavior discussed above is unaltered when using relativistic kinematics (with γv instead of the velocity as a parameter), and also in the case of an analytic (but rapidly varying) potential. The tunneling solutions obtained for the rectangular barrier are limiting solutions of those for a smoothened barrier (e.g. the potential of closely packed point charges). Thus we conclude that both the tunneling phenomenon and the nonuniqueness of physical solutions are general properties of the LD equation, and not artefacts due to unphysical properties of the potential or the non-relativistic approximation.

The key to the physical understanding of these phenomena is the use of the *bound* momentum p_μ introduced in [8]. Apart from the radiation loss (the second term on the r.h.s. of eq.(1)), its rate of change is given by the external force exerted on the electron. If the acceleration is not too large the difference with the "bare" momentum is just a mass renormalization, but when the electron velocity changes rapidly the accompanying self-field needs some time to adjust to the new velocity (the updating is limited by the finite light speed), and p_μ is no longer simply proportional to $\dot{z}_\mu$.

When the electron attacks a steep slope, the bound momentum has to decrease very rapidly. The electron can simply decelerate and bounce back, but if the potential is narrow enough there is a second possibility: the electron can make a "jump", i.e. a short acceleration, during which the bound momentum *decreases*. Because of its *negative* bare mass, the bare electron gives a negative contribution to the bound momentum, which cannot immediately be compensated entirely by the accompanying Coulomb field. When the acceleration ceases, the Coulomb field catches up and the bound momentum increases again, as it should once it reaches the downward slope at the other side of the barrier. In this way tunneling can take place. Note that the kinetic energy $p_0 - m$ becomes negative in the classically forbidden region.

The essential feature of tunneling is that the crossing has to take place in proper times of the order of the pre-acceleration time. For larger widths this could be obtained by considering very high speed electrons, which would effectively see a Lorentz-contracted barrier. It is theoretically not difficult to construct arrangements of individual charges that might show the tunneling phenomenon for very fast electrons[6]. Whereas in some astrophysical applications (for example the motion of charged particles in fields produced by pulsars [9]) there is a combination of fast electron motion with strong fields that necessitates the use of the Lorentz-Dirac equation, it is not clear whether they would provide a testing ground for the tunneling phenomenon described in this paper.

A rough estimate indicates that for phenomena taking place in times of order τ quantum considerations should enter. Since quantum electrodynamics is arguably the most successful physical theory known, it would be interesting to investigate its relation to the tunneling phenomenon discussed in the present paper, and more generally to the Lorentz-Dirac equation. This is outside the scope of the present letter.

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[6]In such cases one should also expect to have to take into account quantum effects.

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